# ON THE SOLUTION OF THE THIRD BOUNDARY PROBLEM FOR AXIALLY-SYMMETRIC REGIONS

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We present a variational method for solving a problem concerning the temperature field distribution inside a two-dimensional axially-symmetric region assuming convective heat transfer on its boundary.

Consider the heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}=-\frac{q_v}{\lambda} \tag{1}$$

for an infinite homogeneous rod with a uniform distribution of heat sources, on whose surface there is given the condition

$$\nabla_n T+\frac{\alpha}{\lambda}T|_\Gamma=0. \tag{2}$$

Putting $u(r, \varphi) = T(r, \varphi)/T^*$, we represent an approximate solution of Eq. (1) in the form

$$u(r,\ \varphi)\simeq\sum_{k=1}^{n}c_k u_k(r,\ \varphi). \tag{3}$$

Using the variational principle due to Ritz, we obtain the following linear system [1] for determining the coefficients $c_k$:

$$\sum_{k=1}^{n}c_k A_{jk}=\frac{q_v}{\lambda}B_j,\quad j=1,\ 2,\ \ldots,\ n. \tag{4}$$

Here

$$A_{jk}=\int_0^{2\pi}d\varphi\int_0^{f(\varphi)}W_{jk}(r,\ \varphi)\,rdr; \tag{5}$$

$$W_{jk}=\frac{\partial u_j}{\partial r}\frac{\partial u_k}{\partial r}+\frac{1}{r^2}\frac{\partial u_j}{\partial \varphi}\frac{\partial u_k}{\partial \varphi}, \tag{6}$$

$$B_j=\int_0^{2\pi}d\varphi\int_0^{f(\varphi)}u_j(r,\ \varphi)\,rdr, \tag{7}$$

where $r_{gr} = f(\varphi)$ is the equation of the contour Γ.

To solve the problem stated above with a direct use of the boundary condition (2) is rather involved since the choice of the system of trial functions is in this case conjugate to an arbitrary combination of first integrals of the partial differential equation corresponding to the given boundary conditions, these integrals, moreover, being of a fairly complicated form [1]. Therefore, we select the set of trial functions $\{u_k\}$ to satisfy, not the boundary condition (2), but a boundary condition of the first kind

$$u|_\Gamma=\theta(\varphi) \tag{8}$$

where the unknown function $\theta(\varphi)$ is defined by an iterational process:

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$$\theta_{\nu+1} = -\frac{\lambda}{\alpha} \nabla_n u^{(\nu)} |_{\Gamma}. \tag{9}$$

Thus as functions satisfying the condition (8) we can take the set

$$u_k(r, \varphi) = 1 - \frac{r^{2k}}{f^{2k}} [1 - \theta(\varphi)], \quad k = 1, 2, \ldots, n. \tag{10}$$

Initially, putting $\alpha = \infty$, we obtain [1]

$$u^{(0)}(r, \varphi) = 1 - \frac{r^2}{f^2(\varphi)}. \tag{11}$$

We may now use the distribution $u^{(0)}(r, \varphi)$ to determine the value of the boundary gradient $\Delta_{\Pi} u^{(0)}$ and thus find the peripheral distribution of $u(r, \varphi)$ in a first approximation:

$$\theta_1(\varphi) = \frac{2\lambda}{\alpha f} \sqrt{1 + \left(\frac{f'}{f}\right)^2}, \tag{12}$$

where the prime indicates differentiation with respect to $\varphi$.

In this approximation the temperature field in a section of the rod is describable by a linear combination of a set of the functions

$$u_k^{(1)}(r, \varphi) = 1 - \frac{r^{2k}}{f^{2k}} [1 - \theta_1(\varphi)]. \tag{13}$$

The temperature $T_1^*$ and the center of the rod may be determined from a balance relationship for a portion of the rod of unit length:

$$q_v S = -\lambda T_1^* \oint_{\Gamma} \nabla_n u^{(1)} dl, \tag{14}$$

where

$$T_1^* = \frac{q_v S}{\lambda \int_0^{2\pi} \left\{ a_n [1 - \theta_1(\varphi)] \left[1 + \left(\frac{f'}{f}\right)^2\right] + b_n \frac{f' \theta_1'}{f[1 - \theta_1(\varphi)]} \right\} d\varphi},$$

$$a_n = 2 \sum_{k=1}^{n} k c_k, \quad b_n = \sum_{k=1}^{n} c_k. \tag{15}$$

We note that in the case of a choice of $u_k(r, \varphi)$ in the form (10)

$$A_{jk} = \frac{1}{2(j+k)} \int_0^{2\pi} \Phi_{jk}(\varphi) d\varphi, \tag{16}$$

$$\Phi_{jk} = 4jk(1 - \theta)^2 + \tau_j \tau_k,$$

$$\tau_j = 2j \frac{f'}{f} (1 - \theta) + \theta', \tag{17}$$

$$B_j = \frac{j}{j+1} S + \frac{1}{2(j+1)} \int_0^{2\pi} \theta(\varphi) f^2(\varphi) d\varphi. \tag{18}$$

If we consider a limiting approach to a cylinder, we can show that $\lim_{n \to \infty} a_n = 2/(1 - (2/\mathrm{Bi})^2)$, where Bi $= \alpha r_0/\lambda$ is the Biot number for a cylinder of radius $r_0$. In addition, $\theta_1 = 2/\mathrm{Bi}$, i.e., the expressions (12) and (15) become exact solutions for $\alpha$ = const.

We remark, in concluding, that the solution of the problem considered here can even be generalized to the case where the heat transfer coefficient $\alpha$ is angle dependent, providing it is symmetric with respect to the coordinate origin selected, since the trial functions in the form (10) have zero derivatives at the point $r = 0$.

## NOTATION

$T^*$ is the temperature at the rod center;
$r$ and $\varphi$ are the polar coordinates;

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| $q_v$ | is the density of the heat release sources; |
| $\lambda$ | is the thermal conductivity of the rod; |
| $f(\varphi)$ | is the distance from the center to the boundary of the region; |
| $\theta(\varphi)$ | is the temperature of the boundary; |
| $\alpha$ | is the heat transfer coefficient; |
| S | is the area of the rod cross section; |
| $\nu$ | is the number of the iteration cycle. |

## LITERATURE CITED

1. L. N. Polyanin, Inzh.-Fiz. Zh., 14, No. 6 (1968).